

KUREPA-TREES AND NAMBA FORCING

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ABSTRACT. We show that compact cardinals and MM are sensitive to λ -closed forcings for arbitrarily large λ . This is done by adding ‘regressive’ λ -Kurepa-trees in either case. We argue that the destruction of regressive Kurepa-trees with MM requires the use of Namba forcing.

1. INTRODUCTION

Say that a tree T of height λ is γ -*regressive* if for all limit ordinals $\alpha < \lambda$ with $\text{cf}(\alpha) < \gamma$ there is a function $f_\alpha : T_\alpha \rightarrow T_{<\alpha}$ which is *regressive*, i.e. $f_\alpha(x) <_T x$ for all $x \in T_\alpha$ and if $x, y \in T_\alpha$ are distinct then $f_\alpha(x)$ or $f_\alpha(y)$ is strictly above the meet of x and y . We give a summary of the main results of this paper:

5 Theorem. *For all uncountable regular λ there is a λ -closed forcing $\mathcal{K}_{\text{reg}}^\lambda$ that adds a λ -regressive λ -Kurepa-tree.*

This is contrasted in Section 4:

7 Theorem. *Assume that κ is a compact cardinal and $\lambda \geq \kappa$ is regular. Then there are no κ -regressive λ -Kurepa-trees.*

Theorems 5 and 7 establish that compact cardinals are sensitive to λ -closed forcings for arbitrarily large λ . This should be compared with the well-known result that a supercompact cardinal κ can be made indestructible by κ -directed-closed forcings [10]. These results drive a major wedge between the notions of λ -closed and λ -directed-closed. Another contrasting known result is that a strong cardinal κ can be made indestructible by κ^+ -closed forcings [3]. In Section 7 we prove

13 Theorem. *Under MM, there are no ω_1 -regressive λ -Kurepa-trees for any uncountable regular λ .*

This shows that MM is sensitive to λ -closed forcings for arbitrarily large λ , thus answering a question from both [7] and [8]. Note that MM is indestructible by ω_2 -directed-closed forcings [8], so again we

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find a remarkable gap between the notions of ω_2 -closed and ω_2 -directed-closed. Interestingly enough though, ω_2 -closed forcings can only violate a very small fragment of MM. To see this, let us denote by Γ_{cov} the class of posets that preserve stationary subsets of ω_1 and have the *covering property*, i.e. every countable set of ordinals in the extension can be covered by a countable set in the ground model. Then we have the following result from [7, p.302]:

1 Theorem. *The axioms PFA, $\text{MA}(\Gamma_{\text{cov}})$ and $\text{MA}^+(\Gamma_{\text{cov}})$ are all indestructible by ω_2 -closed forcings respectively.¹*

So Theorem 5 gives

2 Corollary. *If $\lambda \geq \omega_2$ is regular, then $\text{MA}^+(\Gamma_{\text{cov}})$ is consistent with the existence of a λ -regressive λ -Kurepa-tree.*

Again, compare this with Theorem 13. It is interesting to add that $\text{MA}^+(\Gamma_{\text{cov}})$ in particular implies the axioms PFA^+ and SRP. The typical example of a forcing that preserves stationary subsets of ω_1 but does not have the covering property is Namba forcing and the proofs confirm that Namba forcing plays a crucial role in this context. It has already been established in [9] and [11] that $\text{MA}(\Gamma_{\text{cov}})$ can be preserved in an (ω_1, ∞) -distributive forcing extension in which the Namba-fragment of MM fails. In our case though, the failure of MM is obtained with a considerably milder forcing, i.e. λ -closed for arbitrarily large λ .

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The reader requires a strong background in set-theoretic forcing, a good prerequisite would be [4]. We give some definitions that might not be in this last reference or because we defined them in a slightly different fashion. If Γ is a class of posets then $\text{MA}(\Gamma)$ denotes the statement that whenever $\mathcal{P} \in \Gamma$ and D_ξ ($\xi < \omega_1$) is a collection of dense subsets of \mathcal{P} then there exists a filter G on \mathcal{P} such that $D_\xi \cap G \neq \emptyset$ for all $\xi < \omega_1$. The stronger $\text{MA}^+(\Gamma)$ denotes the statement that whenever $\mathcal{P} \in \Gamma$, D_ξ ($\xi < \omega_1$) are dense subsets of \mathcal{P} , and \dot{S} is a \mathcal{P} -name such that

$$\Vdash_{\mathcal{P}} \dot{S} \text{ is stationary in } \omega_1$$

then there exists a filter G on \mathcal{P} such that $D_\xi \cap G \neq \emptyset$ for all $\xi < \omega_1$, and

$$\dot{S}[G] = \{\gamma < \omega_1 : \exists q \in G (q \Vdash_{\mathcal{P}} \check{\gamma} \in \dot{S})\}$$

is stationary in ω_1 . In particular, PFA is $\text{MA}(\text{proper})$ and MM is $\text{MA}(\text{preserving stationary subsets of } \omega_1)$. The interested reader is referred to [1] and [2] for the history of these *forcing axioms*.

¹See below for a definition of the axioms $\text{MA}(\Gamma)$ and $\text{MA}^+(\Gamma)$.

A partial order is λ -closed if it is closed under descending chains of length less than λ . It is λ -directed-closed if it is closed under directed subsets of size less than λ . [7] proves that PFA is preserved by ω_2 -closed forcings and [8] that MM is preserved by ω_2 -directed-closed forcings.

Namba forcing is denoted by Nm : conditions are trees $t \subseteq \omega_2^{<\omega}$ with a *trunk* $\text{tr}(t)$ such that t is linear below $\text{tr}(t)$ and has splitting \aleph_2 everywhere above the trunk. Smaller trees contain more information. It is known that Namba forcing preserves stationary subsets of ω_1 . If $t \in \text{Nm}$ and $x \in t$ then the last element of x is also called the *tag* of x , denoted as $\text{tag}(x)$, and we define $\text{Suc}_t(x)$ to be the set of tags of all immediate successors of x in t . So $\text{Suc}_t(x)$ is an unbounded subset of ω_2 . In an abuse of notation, a sequence is sometimes confused with its tag. We write $[t]$ for the set of infinite branches through t .

2. STATIONARY LIMITS

For a tree T and an ordinal α , let T_α denote the α th level of T and $T_{<\alpha} = \bigcup_{\xi < \alpha} T_\xi$. If X is a set of ordinals, we write $T \restriction X$ for the subtree $\bigcup_{\xi \in X} T_\xi$. The expression $\text{ht}(T)$ denotes the *height* of T . We only consider trees of functions. If T is a tree and \mathcal{B} a collection of cofinal branches through T then we call \mathcal{B} *non-stationary over T* if there is a function $f : \mathcal{B} \rightarrow T$ which is *regressive*, i.e. $f(b) \in b$ for all $b \in \mathcal{B}$ and if $b, b' \in \mathcal{B}$ are distinct then $f(b)$ or $f(b')$ is strictly above $b \cap b'$. Otherwise we call \mathcal{B} *stationary over T* . A tree T of height κ is called γ -regressive if T_α is non-stationary over $T_{<\alpha}$ for every limit ordinal $\alpha < \kappa$ of cofinality less than γ . The following is easy to check:

3 Remark. Assume that $A \subseteq \alpha$ is cofinal in α . Then T_α is stationary over $T_{<\alpha}$ iff T_α is stationary over $T \restriction A$.

The ω -cofinal limits will figure prominently when dealing with ω_1 -regressive trees, so we prove a useful Lemma about these. For simplicity we only consider trees of height ω . The reader will notice that the following observations are applicable in Section 6. If T is of height ω and \mathcal{B} a collection of infinite branches then for any subset $S \subseteq T$ we let

$$\overline{S} = \{b \in \mathcal{B} : b \cap S \text{ is infinite}\}.$$

If S is countable and \overline{S} uncountable then we call S a *Cantor-subtree* of T . The class $\mathcal{N}(T, \mathcal{B}) \subseteq [H_\theta]^{\aleph_0}$ (for some large enough regular θ) is defined by letting $N \in \mathcal{N}(T, \mathcal{B})$ if and only if there is $b \in \mathcal{B}$ such that $b \subseteq N$ but $b \notin N$. We have the following

4 Lemma. Assume that T has height ω and size \aleph_1 and that \mathcal{B} is a collection of infinite branches. Then the following are equivalent:

- (1) \mathcal{B} is stationary over T .
- (2) (a) Either there is a Cantor-subtree $S \subseteq T$ or
 (b) if we identify T with ω_1 by any enumeration then

$$E_{\mathcal{B}} = \{\alpha < \omega_1 : \sup(b) = \alpha \text{ for some } b \in \mathcal{B}\}$$

is stationary in ω_1 .

- (3) $\mathcal{N}(T, \mathcal{B})$ is stationary in $[H_\theta]^{\aleph_0}$.

Proof. The equivalence of (1) and (3) can be found in [6, p.112] and the implication (2) \implies (1) is easy.

For (3) \implies (2), assume $\neg(2)$ and show $\neg(3)$: pick an enumeration $e : \omega_1 \rightarrow T$ such that $E_{\mathcal{B}}$ is nonstationary if we identify nodes with countable ordinals via the enumeration e . Pick a structure $N \prec H_\theta$ such that $e, T, \mathcal{B} \in N$ and set $\gamma = N \cap \omega_1$, so we have $\gamma \notin E_{\mathcal{B}}$. Let $b \in \mathcal{B}$ be such that $b \subseteq N$. Then $\sup(b) < \gamma$ holds. Now define

$$\mathcal{A} = \{c \in \mathcal{B} : \sup(c) = \sup(b)\}.$$

Note that $\mathcal{A} \in N$ and \mathcal{A} is countable since we know by $\neg(2)(a)$ that $\overline{\sup(b)}$ is countable. So $\mathcal{A} \subseteq N$, therefore $b \in N$. This shows that $N \notin \mathcal{N}(T, \mathcal{B})$ and $\mathcal{N}(T, \mathcal{B})$ is non-stationary. \square

Note that the equivalence of (1) and (3) is to some extent already in [1, p.955] but our result differs slightly from this last reference as we have a stronger notion of non-stationarity. See also [6] for variations of Lemma 4 in uncountable heights.

3. CREATING REGRESSIVE KUREPA-TREES

Let λ be a regular uncountable cardinal throughout this section. We describe the natural forcing $\mathcal{K}_{\text{reg}}^\lambda$ to add a λ -regressive λ -Kurepa-tree and show that this forcing is λ -closed. We may assume the cardinal arithmetic $2^{<\lambda} = \lambda$, otherwise a preliminary Cohen-subset of λ could be added. Conditions of $\mathcal{K}_{\text{reg}}^\lambda$ are pairs (T, h) , where

- (1) T is a tree of height $\alpha + 1$ for some $\alpha < \lambda$ and each level has size $< \lambda$.
- (2) T is λ -regressive, i.e. if $\xi \leq \alpha$ then T_ξ is non-stationary over $T_{<\xi}$.
- (3) $h : T_\alpha \longrightarrow \lambda^+$ is 1-1.

The condition (T, h) is stronger than (S, g) if

- $S = T \upharpoonright \text{ht}(S)$.
- $\text{rng}(g) \subseteq \text{rng}(h)$.
- $g^{-1}(\nu) \leq_T h^{-1}(\nu)$ for all $\nu \in \text{rng}(g)$.

A generic filter G for $\mathcal{K}_{\text{reg}}^\lambda$ will produce a λ -regressive λ -tree T_G in the first coordinate and the sets

$$b_\nu = \{x \in T_G : \text{there is } (T, h) \in G \text{ such that } h(x) = \nu\}$$

for $\nu < \lambda^+$ form a collection of λ^+ -many mutually different λ -branches through the tree T_G . Notice also that the standard arguments for λ^+ -cc go through here as we assumed $2^{<\lambda} = \lambda$.

So we are done once we show that $\mathcal{K}_{\text{reg}}^\lambda$ is λ -closed. To this end, let (T^ξ, h^ξ) ($\xi < \gamma$) be a descending chain of conditions of length less than λ . We can obviously assume that γ is a limit ordinal. If the height of T^ξ is $\alpha^\xi + 1$, let $\alpha^\gamma = \sup_{\xi < \gamma} \alpha^\xi$. We want to extend the tree

$$T^* = \bigcup_{\xi < \gamma} T^\xi,$$

so we have to define the α^γ th level: whenever $\nu \in \text{rng}(h^\xi)$ for some $\xi < \gamma$, then there is exactly one α^γ -branch c_ν that has color ν on a final segment. Now define

$$T_{\alpha^\gamma}^\gamma = \{c_\nu : \nu \in \text{rng}(h^\xi) \text{ for some } \xi < \gamma\}$$

and let T^γ be the tree T^* with the level $T_{\alpha^\gamma}^\gamma$ on top. The 1-1 function $h^\gamma : T_{\alpha^\gamma}^\gamma \rightarrow \lambda^+$ is defined by letting

$$h^\gamma(c_\nu) = \nu.$$

We claim that (T^γ, h^γ) is a condition: the only thing left to check is that $T_{\alpha^\gamma}^\gamma$ is non-stationary over T^* . But this is witnessed by the function

$$f(c_\nu) = \text{the } <_T\text{-least } x \in c_\nu \text{ such that there is } \xi < \gamma \text{ with } h^\xi(x) = \nu.$$

Notice that f is regressive: if

$$f(c_\nu) \leq_T f(c_\mu) \leq_T c_\nu \cap c_\mu,$$

let ξ witness that $f(c_\mu) = x$, i.e. $h^\xi(x) = \mu$. Then $h^\xi(x)$ must be color ν as well since $f(c_\nu) \leq_T x$ has color ν . Thus, $\nu = h^\xi(x) = \mu$.

But (T^γ, h^γ) extends the chain (T^ξ, h^ξ) ($\xi < \gamma$), so we just showed

5 Theorem. $\mathcal{K}_{\text{reg}}^\lambda$ is a λ -closed forcing that adds a λ -regressive λ -Kurepa-tree.

We emphasize again that the forcing $\mathcal{K}_{\text{reg}}^\lambda$ is *not* ω_2 -directed-closed but the reader can check that the usual forcing to add a plain λ -Kurepa-tree (see e.g. [4]) actually *is* λ -directed-closed.

4. DESTROYING REGRESSIVE KUREPA-TREES ABOVE A COMPACT CARDINAL

If λ is a regular uncountable cardinal then a tree T is called a *weak λ -Kurepa-tree* if

- T has height λ ,
- each level has size $\leq \lambda$ and
- T has λ^+ -many cofinal branches.

6 Lemma. *Suppose that λ is a regular uncountable cardinal and there is an elementary embedding $j : V \longrightarrow M$ such that $\eta = \sup(j''\lambda) < j(\lambda)$ and $\text{cf}^M(\eta) < j(\kappa)$. Then there are no κ -regressive weak λ -Kurepa-trees.*

Proof. Suppose that T is a κ -regressive weak λ -Kurepa-tree and j as above. Then there is a regressive function f_η defined on the level $(jT)_\eta$. If b is a cofinal branch through T , then we find $\alpha_b < \lambda$ such that

$$f_\eta(jb \restriction \eta) \leq_{jT} jb \restriction j(\alpha_b) = j(b \restriction \alpha_b).$$

Note that if b and b' are two distinct branches through T then jb and jb' must disagree below η . Moreover, $j(b \restriction \alpha_b) \neq j(b' \restriction \alpha_{b'})$ holds because f_η is regressive. Then the assignment $b \longmapsto b \restriction \alpha_b$ must be 1-1, which is a contradiction to the fact that T has λ^+ -many branches. \square

Recall that a cardinal κ is λ -compact if there is a fine ultrafilter on $\mathcal{P}_\kappa\lambda$. If λ is regular, the elementary embedding $j : V \longrightarrow M$ with respect to such a fine ultrafilter has the following properties:

- the critical point of j is κ ,
- there is a discontinuity at λ , i.e. $\eta = \sup(j''\lambda) < j(\lambda)$ and
- $\text{cf}^M(\eta) < j(\kappa)$.

(see [5, §22] for more details). A cardinal κ is said to be *compact* if it is λ -compact for all λ , so it follows from Lemma 6 and the above definition:

7 Theorem. *Assume that κ is a compact cardinal and $\lambda \geq \kappa$ is regular. Then there are no κ -regressive weak λ -Kurepa-trees.*

Using Theorem 5, we have

8 Corollary. *Compact cardinals are sensitive to λ -closed forcings for arbitrarily large λ .*

It was known before that adding a slim² κ -Kurepa-tree destroys the ineffability of κ and that slim κ -Kurepa-trees can be added with κ -closed forcing. But note that our notion of regressive is more universal: slim Kurepa-trees can exist above compact or even supercompact cardinals.

5. OSCILLATING BRANCHES

Now assume that T is an ω_2 -tree: we enumerate each level by letting

$$(5.1) \quad T_\alpha = \{\tau(\alpha, \xi) : \xi < \omega_1\} \text{ for all } \alpha < \omega_2.$$

In this situation we identify branches with functions from ω_2 to ω_1 that are induced by the enumerations of the levels. If $A \subseteq \omega_2$ is unbounded and $b : \omega_2 \rightarrow \omega_1$ is an ω_2 -branch through T then we say that b *oscillates on* A if for all $\alpha < \omega_2$ and all $\zeta < \omega_1$ there is $\beta > \alpha$ in A and $\xi > \zeta$ such that $b(\beta) = \xi$.

9 Lemma. *Assume that T is an ω_2 -Kurepa-tree with an enumeration $\tau(\alpha, \xi)$ ($\alpha < \omega_2$, $\xi < \omega_1$) as in (5.1) and A_ι ($\iota < \omega_2$) are \aleph_2 -many unbounded subsets of ω_2 . Then there is an ω_2 -branch b through T that oscillates on every A_ι ($\iota < \omega_2$).*

Proof. Assume not, then for every ω_2 -branch b there is $\iota_b < \omega_2$ and there are $\alpha_b < \omega_2$, $\zeta_b < \omega_1$ such that

$$b \upharpoonright (A_{\iota_b} \setminus \alpha_b) \subseteq \{\tau(\alpha, \xi) : \alpha \in A_{\iota_b} \setminus \alpha_b, \xi < \zeta_b\}.$$

By a cardinality argument we can find \aleph_3 -many branches b such that $\iota_0 = \iota_b$, $\alpha_0 = \alpha_b$ and $\zeta_0 = \zeta_b$. But then each of these branches is a different branch through the tree

$$T_0 = \{\tau(\alpha, \xi) : \alpha \in A_{\iota_0} \setminus \alpha_0, \xi < \zeta_0\}.$$

T_0 has countable levels but \aleph_3 -many branches, a contradiction. □

6. DESTROYING REGRESSIVE KUREPA-TREES WITH MM

We introduce a simplified notation for the following arguments: if $f : t \rightarrow \omega_1$ for some $t \in \text{Nm}$ and $\pi \in [t]$ then we let

$$\sup^{(f)}(\pi) = \sup_{n < \omega} f(\pi \upharpoonright n).$$

If $b : \omega_2 \rightarrow \omega_1$ is an ω_2 -branch and $x \in \omega_2^{<\omega}$ then $b(x)$ really denotes the countable ordinal $b(\text{tag}(x))$.

²A κ -Kurepa-tree T is called *slim* if $|T_\alpha| \leq |\alpha|$ for all $\alpha < \kappa$.

10 Lemma. *Assume that T is an ω_2 -Kurepa-tree and \mathcal{B} is the set of branches. Let $\tau(\alpha, \xi)$ ($\alpha < \omega_2$, $\xi < \omega_1$) be an enumeration as in (5.1). Then in the Namba extension V^{Nm} there is a sequence*

$$\Delta_G = \langle \delta_n^G : n < \omega \rangle$$

cofinal in ω_2^V such that

$$\dot{E}_{\mathcal{B}} = \{\sup^{(b)}(\Delta_G) : b \in \mathcal{B}\}$$

is stationary relative to every stationary $S \subseteq \omega_1$ in V , i.e. $\dot{E}_{\mathcal{B}} \cap S$ is stationary for all stationary $S \subseteq \omega_1$ in the ground model.

Proof. Assume that \dot{C} is an Nm-name for a club in ω_1 , $S \subseteq \omega_1$ is a stationary set in V and t_0 a condition in Nm. Our goal is to find a condition $t_3 \leq t_0$ and an ordinal $\xi_0 \in S$ such that $t_3 \Vdash \xi_0 \in \dot{C} \cap \dot{E}_{\mathcal{B}}$. By a fusion argument similar to the ones in [11, p.188], we construct a condition $t_1 \leq t_0$ and a coloring $f : t_1 \rightarrow \omega_1$ such that

- (1) f increases on chains, i.e. if $v \subsetneq x$ are elements of t_1 then $f(v) < f(x)$.
- (2) if the height of x in t_1 is odd and x is above the trunk then there is $\zeta < \omega_1$ such that

$$|\{x^\frown \beta \in t_1 \mid f(x^\frown \beta) = \xi\}| = \aleph_2 \text{ for all } \xi > \zeta,$$

i.e. each ordinal in a final segment of ω_1 has \aleph_2 -many preimages in the set $\text{Suc}_{t_1}(x)$.

- (3) if $G \subseteq \text{Nm}$ is generic with $t_1 \in G$ and $\pi : \omega \rightarrow \omega_2^V$ is the corresponding Namba-sequence then $\sup^{(f)}(\pi) \in \dot{C}[G]$.

Given the condition t_1 , we apply Lemma 9 to find a branch b that oscillates on all sets $\text{Suc}_{t_1}(x)$ ($x \in t_1$). Using (1) and (2), we thin out again to get a condition $t_2 \leq t_1$ with the following property:

- (4) if $v \subsetneq x \subsetneq y$ is a chain in t_2 above the trunk and the height of x is odd then $f(v) < b(x) < f(y)$.

Note that in particular (2) can be preserved by passing to the condition t_2 , so we may assume that t_2 has properties (1)-(4). Let us also assume for notational simplicity that the height of $\text{tr}(t_2)$ is even. The next step is to find $t_3 \leq t_2$ and $\xi_0 \in S$ such that

- (5) $\sup^{(f)}(\pi) = \xi_0$ for all branches π in $[t_3]$.

To find t_3 and ξ_0 , we define a game $\mathbb{G}(\gamma)$ for every limit $\gamma < \omega_1$. Fix a ladder sequence $l(\gamma) = (\gamma_n : n < \omega)$ for each such γ . The game $\mathbb{G}(\gamma)$ is played as follows:

I	α_0	α_1	α_2	α_3	\dots
II	β_0	β_1	β_2	β_3	\dots

where for all $n < \omega$

- $\alpha_n < \beta_n < \omega_2$,
- $s_n = \text{tr}(t_2)^\frown(\beta_i : i \leq n) \in t_2$ and
- $f(s_n) \in (\gamma_n, \gamma)$ whenever n is even.

II wins if he can make legal moves at each step, so the game is determined.

10.1 Claim. *II wins $\mathbb{G}(\gamma)$ for club many γ 's.*

Proof. Assume not, then there is a stationary $U \subseteq \omega_1$ such that player I wins $\mathbb{G}(\gamma)$ for each $\gamma \in U$ via the strategy σ_γ . Now pick a countable elementary N such that $\xi = N \cap \omega_1 \in U$ and $t_2, f, l, U \in N$.

A ladder sequence $l(\xi) = (\xi_n : n < \omega)$ converging to ξ is given and we define a sequence $(\beta_n : n < \omega)$ inductively as follows: let β_n be the least

$$\beta > \sup_{\gamma \in U} \sigma_\gamma(\beta_i : i < n)$$

such that

- $s = \text{tr}(t_2)^\frown(\beta_i : i < n)^\frown \beta \in t_2$ and
- $f(s) \in (\xi_n, \xi)$ whenever n is even.

Such a β exists in N by (2) and elementarity. Note that $(\beta_n : n < \omega)$ is a possible record of moves for player II if player I goes along with the strategy σ_ξ . But II obviously wins the game $\mathbb{G}(\xi)$ if the sequence $(\beta_n : n < \omega)$ is played, a contradiction. This proves the claim. \square

Given the claim, pick $\xi_0 \in S$ above all $b(x)$ ($x \subseteq \text{tr}(t_2)$) such that II wins the game $\mathbb{G}(\xi_0)$. Now we can easily find a condition $t_3 \leq t_2$ with property (5).

If we fix a generic $G \subseteq \text{Nm}$ with $t_3 \in G$ and let $\pi_G : \omega \longrightarrow \omega_2^V$ be the corresponding Namba-sequence, we can define $\delta_n^G = \pi_G(2n + 1)$ and $\Delta_G = \langle \delta_n^G : n < \omega \rangle$. Then we have

- (6) $\sup^{(f)}(\Delta_G) \in \dot{C}[G]$ by (3),
- (7) $\sup^{(f)}(\Delta_G) = \xi_0$ by (5) and
- (8) $\sup^{(f)}(\Delta_G) = \sup_{n < \omega} b(\delta_n^G) = \sup^{(b)}(\Delta_G)$ by (4).

But this finishes the proof since

$$\xi_0 \in \dot{C}[G] \cap \dot{E}_B[G] \cap S.$$

\square

11 Corollary. *Assume that T is an ω_2 -Kurepa-tree and \mathcal{B} the set of ground model branches through T . Then \mathcal{B} is stationary over T in the Namba extension.*

Finally we get the main result for ω_2 . We will prove a more general version of this in Theorem 13.

12 Theorem. *There are no ω_1 -regressive ω_2 -Kurepa-trees under MM.*

Proof. Assume that T is an ω_1 -regressive ω_2 -Kurepa-tree and that

$$\tau(\alpha, \xi) \ (\alpha < \omega_2, \xi < \omega_1)$$

is an enumeration as in (5.1). Look at the iteration $\mathbb{P} = \text{Nm} * \text{CS}(\dot{E}_{\mathcal{B}})$, where $\text{CS}(\dot{E}_{\mathcal{B}})$ shoots a club through the set $\dot{E}_{\mathcal{B}}$ from the statement of Lemma 10. The poset \mathbb{P} preserves stationary subsets of ω_1 by the fact that $\dot{E}_{\mathcal{B}}$ is stationary relative to every stationary set in V . But we have that $\dot{E}_{\mathcal{B}}$ is club in $V^{\mathbb{P}}$, so we can use MM to get a sequence $\Delta = \langle \delta_n : n < \omega \rangle$ converging to $\delta < \omega_2$ such that

$$\{\sup^{(b)}(\Delta) : b \text{ is a } \delta\text{-sequence in } T_{\delta}\}$$

is club in ω_1 . Using Lemma 4, we see that T_{δ} is definitely stationary over $T \restriction \Delta$. So T_{δ} is stationary over $T_{<\delta}$ by Remark 3. Since $\text{cf}(\delta) = \omega$, this contradicts the fact that T is ω_1 -regressive. \square

7. LARGER HEIGHTS

Starting from Theorem 12, we generalize the result to weak Kurepa-trees in all uncountable regular heights.

13 Theorem. *Under MM, there are no ω_1 -regressive weak λ -Kurepa-trees for any uncountable regular λ .*

Proof. Since PFA destroys weak ω_1 -Kurepa-trees (see [1]), we may assume that λ is at least ω_2 . Now assume that T is an ω_1 -regressive weak λ -Kurepa-tree and let $\mathcal{P} = \text{Col}(\omega_2, \lambda)$ be the usual ω_2 -directed collapse. Note that \mathcal{P} has the λ^+ -cc, because $\lambda^{\omega_1} = \lambda$ holds under MM (see [2]). So the tree T has a cofinal subtree T^* in $V^{\mathcal{P}}$ that is an ω_1 -regressive weak ω_2 -Kurepa-tree. By throwing away some nodes if necessary, we may assume that T^* has the property that

$$(7.1) \quad T_x^* = \{y \in T^* : x \leq_T y\} \text{ has } \aleph_3\text{-many branches for all } x \in T^*.$$

Now we define an ω_2 -directed forcing \mathcal{Q} in $V^{\mathcal{P}}$ that shoots an actual ω_2 -Kurepa-subtree through the tree T^* : conditions of \mathcal{Q} are pairs of the form (S, B) , where

- (1) S is a downward-closed subtree of T^* of height $\alpha + 1$ for some ordinal $\alpha < \omega_2$,
- (2) $|S| \leq \omega_1$,
- (3) B is a nonempty set of branches cofinal in T^* and $|B| \leq \aleph_1$,
- (4) $b \restriction (\alpha + 1) \subseteq S$ for all $b \in B$.

We let $(S_0, B_0) \geq_{\mathcal{Q}} (S_1, B_1)$ if $S_0 = S_1 \upharpoonright \text{ht}(S_0)$ and $B_0 \subseteq B_1$.

If $X \subseteq \mathcal{Q}$ is a set of mutually compatible conditions of size $\leq \aleph_1$ then we let S_X and B_X be the unions over the first respectively second coordinates of X . Now S_X can be end-extended to a tree \bar{S}_X of successor height by extending at least the cofinal branches in the non-empty set B_X . But then (\bar{S}_X, B_X) is a condition stronger than every condition in X , hence \mathcal{Q} is ω_2 -directed-closed. An easy cardinality argument shows that \mathcal{Q} has the \aleph_3 -cc because $2^{\aleph_1} = \aleph_2$ holds in $V^{\mathcal{P}}$. It is now straightforward that a generic filter $H \subseteq \mathcal{Q}$ will produce an ω_2 -tree in the first coordinate which is ω_1 -regressive since it is a subtree of the original tree T and notice that \mathcal{P} and \mathcal{Q} both preserve uncountable cofinalities. On the other hand, a density argument using (7.1) shows that the set

$$\mathfrak{B} = \bigcup \{B : \text{there is } S \text{ such that } (S, B) \in H\}$$

has cardinality \aleph_3 , so H induces an ω_1 -regressive ω_2 -Kurepa-tree. The composition of two ω_2 -directed-closed forcings is again ω_2 -directed-closed and it was mentioned in the introduction that ω_2 -directed-closed forcings preserve MM, so we have the situation:

- $V^{\mathcal{P} * \mathcal{Q}} \models \text{MM}$
- $V^{\mathcal{P} * \mathcal{Q}} \models \text{“there is an } \omega_1\text{-regressive } \omega_2\text{-Kurepa-tree.”}$

But this contradicts Theorem 12. □

Using Theorem 5, we have

14 Corollary. *MM is sensitive to λ -closed forcing algebras for arbitrarily large λ .*

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